

Normal Cumulative Distribution Function (CDF)

1. Suppose that an automobile muffler is designed so that its lifetime (in months) is approximately normally distributed with mean 26 months and standard deviation 4 months. The manufacturer has decided to use a marketing strategy in which the muffler is covered by warranty for 18 months. Approximately what proportion of the mufflers will fail before the warranty expires?

Solution: Let X be the lifetime of a typical muffler in months; this is normally distributed with mean $\mu = 26$ and standard deviation $\sigma = 4$. The muffler will fail before the warranty expires if and only if $X < 18$. Thus, the proportion of the mufflers that fail before the warranty expires is equal to the probability that X is less than 18. Define $Z = (X - \mu)/\sigma$, a standard normal random variable. Then,

$$\begin{aligned} P(X < 18) &= P\left(\frac{X - \mu}{\sigma} < \frac{18 - 26}{4}\right) \\ &= P(Z < -2) \\ &= \Phi(-2) \\ &= .02275. \end{aligned}$$

2. Suppose that the daily demand for change (meaning coins) in a particular store is approximately normally distributed with mean \$800.00 and standard deviation \$60.00. What is the probability that, on any particular day, the demand for change will be below \$600?

Solution: Let X be the demand for change on a particular day (in dollars); this is a normal random variable with mean $\mu = 800$ and standard deviation $\sigma = 60$. Now

$$P(X < 600) = P\left(\frac{X - \mu}{\sigma} < \frac{600 - 800}{60}\right) = \Phi(-3.3) = .0004834.$$

Inverse Normal CDF

3. Suppose that the daily demand for change (meaning coins) in a particular store is approximately normally distributed with mean \$800.00 and standard deviation \$60.00. Find the amount M of change to keep on hand if one wishes, with certainty 99%, to have enough change. That is, find M so that $P(X \leq M) = 0.99$.

Solution: We have

$$.99 = P(X \leq M) = P\left(\frac{X - \mu}{\sigma} \leq \frac{M - 800}{60}\right) = \Phi\left(\frac{M - 800}{60}\right).$$

Thus,

$$\frac{M - 800}{60} = \Phi^{-1}(.99).$$

Consulting the normal inverse CDF table, we see that $\Phi(2.3263) = .99$ and We take $\Phi^{-1}(.99) = 2.3263$, so that

$$M = 800 + 60 \times 2.3263 = 939.578.$$

4. Suppose that Z is a standard normal random variable. Find the value w so that $P(|Z| \leq w) = 0.60$.

Solution: The problem is asking for w such that $P(-w \leq Z \leq w) = 0.60$. We can look up this value directly, in the fourth column, getting $w = .8416$.

For an alternative solution, note that

$$P(Z < -w) + P(-w \leq Z \leq w) + P(Z > w) = 1.$$

Also, $P(Z < -w) = P(Z > w)$ (draw a picture if this is not obvious to you). In this case, we must have that $P(Z < -w) = 0.20$. Therefore, $P(Z < w) = 0.80$.

Now, $\Phi(w) = 0.80$. The normal CDF table tells us that $\Phi(0.8416) = 0.80$; we take $w = 0.8416$.

5. A machine that dispenses corn flakes into packages provides amounts that are approximately normally distributed with mean weight 20 ounces and standard deviation 0.6 ounce. Suppose that the weights and measures law under which you must operate allows you to have only 5% of your packages under the weight stated on the package. What weight should you print on the package?

Solution: Let X be the weight in ounces of a typical package; this is approximately normally distributed with mean $\mu = 20$ and standard deviation $\sigma = 0.6$. We seek a printed weight, w , such that $P(X < w) = .05$. Define $Z = (X - \mu)/\sigma$, a standard normal random variable. We have the following relation:

$$.05 = P(X < w) = P\left(\frac{X - \mu}{\sigma} < \frac{w - 20}{0.6}\right) = P\left(Z < \frac{w - 20}{0.6}\right) = \Phi\left(\frac{w - 20}{0.6}\right).$$

Thus,

$$\frac{w - 20}{0.6} = \Phi^{-1}(.05).$$

With a normal table, we compute $\Phi(-1.6449) = 0.05$. Finally,

$$\frac{w - 20}{0.6} = -1.6449,$$

so $w = 20 - 0.6 \times 1.6449 = 19.01306$. We would probably round this to 19 and print “19 ounces” on the box.

The Central Limit Theorem

6. Consider the population of all Fortune 500 CEOs and their salaries. Suppose that the mean salary (in millions of dollars) is $\mu = 20$, and the standard deviation of the salaries is $\sigma = 5$. You sample 50 CEOs and find their salaries.
- (a) Draw a histogram of what you think the population looks like.

- (b) Consider the sample mean \bar{X} to be a random variable. What is the expectation of \bar{X} ?

Solution: $E[\bar{X}] = \mu = 20$.

- (c) What is the standard deviation of \bar{X} ?

Solution: $sd[\bar{X}] = \frac{\sigma}{\sqrt{n}} = \frac{5}{\sqrt{50}} \approx .707$.

- (d) Draw a picture of what you think the PDF of \bar{X} looks like.

Solution: Normal with mean 20, standard deviation .707.

7. You draw a random sample of size $n = 64$ from a population with mean $\mu = 50$ and standard deviation $\sigma = 16$. From this, you compute the sample mean, \bar{X} .

(a) What are the expectation and standard deviation of \bar{X} ?

Solution:

$$\begin{aligned} E[\bar{X}] &= \mu = 50, \\ \text{sd}[\bar{X}] &= \frac{\sigma}{\sqrt{n}} = \frac{16}{\sqrt{64}} = 2. \end{aligned}$$

(b) Approximately what is the probability that the sample mean is above 54?

Solution: The sample mean has expectation 50 and standard deviation 2. By the central limit theorem, the sample mean is approximately normally distributed. Thus, by the empirical rule, there is roughly a 2.5% chance of being above 54 (2 standard deviations above the mean).

(c) Do you need any additional assumptions for part (c) to be true?

Solution: No. Since the sample size is large ($n \geq 30$), the central limit theorem applies.

8. You draw a random sample of size $n = 16$ from a population with mean $\mu = 100$ and standard deviation $\sigma = 20$. From this, you compute the sample mean, \bar{X} .

(a) What are the expectation and standard deviation of \bar{X} ?

Solution:

$$\begin{aligned} E[\bar{X}] &= \mu = 100, \\ \text{sd}[\bar{X}] &= \frac{\sigma}{\sqrt{n}} = \frac{20}{\sqrt{16}} = 5. \end{aligned}$$

(b) Approximately what is the probability that the sample mean is between 95 and 105?

Solution: The sample mean has expectation 100 and standard deviation 5. If it is approximately normal, then we can use the empirical rule to say that there is a 68% of being between 95 and 105 (within one standard deviation of its expectation).

(c) Do you need any additional assumptions for part (c) to be true?

Solution: Yes, we need to assume that the population is normal. The sample size is small ($n < 30$), so the central limit theorem may not be in force.