The Central Limit Theorem – Solutions

STAT-UB.0103 – Statistics for Business Control and Regression Models

The Central Limit Theorem

1. Consider the population of all Fortune 500 CEOs and their salaries. Suppose that the mean salary (in millions of dollars) is $\mu = 20$, and the standard deviation of the salaries is $\sigma = 5$. You sample 50 CEOs and find their salaries.

   (a) Draw a histogram of what you think the population looks like.

   (b) Consider the sample mean $\bar{X}$ to be a random variable. What is the expectation of $\bar{X}$?

   Solution: $\mathrm{E}[\bar{X}] = \mu = 20$.

   (c) What is the standard deviation of $\bar{X}$?

   Solution: $\mathrm{sd}[\bar{X}] = \frac{\sigma}{\sqrt{n}} = \frac{5}{\sqrt{50}} \approx .707$.

   (d) Draw a picture of what you think the PDF of $\bar{X}$ looks like.

   Solution: Normal with mean 20, standard deviation .707.

2. You draw a random sample of size $n = 64$ from a population with mean $\mu = 50$ and standard deviation $\sigma = 16$. From this, you compute the sample mean, $\bar{X}$.

   (a) What are the expectation and standard deviation of $\bar{X}$?

   Solution:

   $$\mathrm{E}[\bar{X}] = \mu = 50,$$
   $$\mathrm{sd}[\bar{X}] = \frac{\sigma}{\sqrt{n}} = \frac{16}{\sqrt{64}} = 2.$$

   (b) Approximately what is the probability that the sample mean is above 54?

   Solution: The sample mean has expectation 50 and standard deviation 2. By the central limit theorem, the sample mean is approximately normally distributed. Thus, by the empirical rule, there is roughly a 2.5% chance of being above 54 (2 standard deviations above the mean).

   (c) Do you need any additional assumptions for part (c) to be true?

   Solution: No. Since the sample size is large ($n \geq 30$), the central limit theorem applies.

3. You draw a random sample of size $n = 16$ from a population with mean $\mu = 100$ and standard deviation $\sigma = 20$. From this, you compute the sample mean, $\bar{X}$.

   (a) What are the expectation and standard deviation of $\bar{X}$?

   Solution:

   $$\mathrm{E}[\bar{X}] = \mu = 100,$$
   $$\mathrm{sd}[\bar{X}] = \frac{\sigma}{\sqrt{n}} = \frac{20}{\sqrt{16}} = 5.$$

   (b) Approximately what is the probability that the sample mean is between 95 and 105?

   Solution: The sample mean has expectation 100 and standard deviation 5. If it is approximately normal, then we can use the empirical rule to say that there is a 68% of being between 95 and 105 (within one standard deviation of its expecation).

   (c) Do you need any additional assumptions for part (c) to be true?

   Solution: Yes, we need to assume that the population is normal. The sample size is small ($n < 30$), so the central limit theorem may not be in force.