Conditional Probability – Solutions

STAT-UB.0103 – Statistics for Business Control and Regression Models

Counting (Review)

1. There are 10 people in a club. How many ways are there to choose the following:

   (a) A president, vice president, and treasurer?

   Solution:
   $$10 \cdot 9 \cdot 8 = 720$$

   (b) A 3-person committee.

   Solution:
   $$\binom{10}{3} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120.$$

Conditional Probability

2. Here is a table tabulated frequencies for the majors and genders of the students in the class survey.

	Gender		
Major	Female	Male	Total
Finance	12	20	32
Other	4	3	7
Undecided	10	15	25
Total	26	38	64

   (a) List 2 interesting conclusions you can draw from the data in the table. At least one of your conclusions should compare Females and Males. All of your conclusions should involve proportions or probabilities.
   Example: $\frac{32}{64} = 50\%$ *of students listed Finance as their major.*

   Solution:

   - Roughly the same proportions of Females and Males listed Finance as their major ($\frac{12}{26} = 46\%$ for Female; $\frac{20}{38} = 52\%$ for Males).
   - $\frac{20}{32} = 62.5\%$ of the students who listed Finance are Male.

- $\frac{25}{64} = 39\%$ of students listed major as Undecided ($\frac{10}{26} = 38\%$ of Females; $\frac{15}{38} = 39\%$ of Males).
- $\frac{15}{25} = 60\%$ of the students who listed their major as Undecided are Male.
- $\frac{4}{26} = 15\%$ of Females listed their major as Other, while only $\frac{3}{38} = 8\%$ of Males did.
- $\frac{4}{7} = 57\%$ of the students who listed their major as Other are Female.

(Other answers are also valid.)

(b) Which of the conclusions you listed are conditional probabilities?

Solution:

- $P(\text{Finance} \mid \text{Female}) = \frac{12}{26}$
- $P(\text{Finance} \mid \text{Male} = \frac{20}{38}$
- $P(\text{Male} \mid \text{Finance}) = \frac{20}{32}$
- $P(\text{Undecided} \mid \text{Female} = \frac{10}{26}$
- $P(\text{Undecided} \mid \text{Male} = \frac{15}{38}$
- $P(\text{Male} \mid \text{Undecided}) = \frac{15}{25}$
- $P(\text{Other} \mid \text{Female}) = \frac{4}{26}$
- $P(\text{Other} \mid \text{Male}) = \frac{3}{38}$
- $P(\text{Female} \mid \text{Other}) = \frac{4}{7}$

(Other answers are also valid.)

3. The following table shoes the number of pairs of shoes reported and the genders of the 63 survey respondents who answered the questions "How many pairs of shoes do you own?".

	Pairs of Shoes					
Gender	1–3	4–6	7–9	10–12	>12	Total
Female	0	1	4	9	11	25
Male	3	13	11	7	4	38
Total	3	14	15	16	15	63

Use this data to answer the following questions.

(a) If we pick a random survey respondent out of these 63, and it turns out that the respondent happens to be Female, what is the chance that the respondent will have 10–12 pairs of shoes.

Solution:

$$\begin{aligned} P(\text{10–12 Shoes} \mid \text{Female}) &= \frac{9}{25} \\ &= 36\%. \end{aligned}$$

(b) Answer the previous problem using the equation $P(B \mid A) = \frac{P(A \cap B)}{P(A)}$. Here, the events are as follow:

$$\begin{aligned} A &= \text{respondent is female}, \\ B &= \text{respondent owns 10–12 pairs of shoes}. \end{aligned}$$

Solution:

$$\begin{aligned} P(A) &= \frac{25}{63} \\ P(A \cap B) &= \frac{9}{63} \\ P(B \mid A) &= \frac{P(A \cap B)}{P(A)} \\ &= \frac{(9/63)}{(25/63)} \\ &= \frac{9}{25}. \end{aligned}$$

(c) If we pick a respondent who owns 4–6 pairs of shoes, what is the chance that the respondent is male?

Solution:

$$P(\text{Male} \mid \text{4–6 pairs of shoes}) = \frac{13}{14} = 93\%.$$

(d) Find the conditional probability $P(>12 \text{ pairs of shoes } \mid \text{Female})$.

Solution:

$$P(>12 \text{ pairs of shoes } \mid \text{Female}). = \frac{11}{25} = 44\%.$$

(e) Find the conditional probability $P(\text{Female} \mid >12 \text{ pairs of shoes})$.

Solution:

$$P(\text{Female} \mid >12 \text{ pairs of shoes}) = \frac{11}{15} = 73\%.$$

(f) Explain, the difference between (d) and (e).

Solution: $P(>12 \text{ pairs of shoes } \mid \text{Female})$ is the proportion of female survey respondents who own more than 12 pairs of shoes; $P(\text{Female} \mid >12 \text{ pairs of shoes})$ is the proportion of respondents with more than 12 pairs of shoes who are female.

The Multiplicative Rule

4. Out of the 70 students enrolled in the class, 25 are female (36%) and 45 are male (64%). Suppose that we randomly select two different students.

   (a) What is the probability that both students are male?

   Solution: Define the two events

   $$\begin{aligned} A &= \text{the first student picked is male} \\ B &= \text{the second student picked is male.} \end{aligned}$$

   Then, $P(A) = \frac{45}{70}$, and $P(B \mid A) = \frac{44}{69}$. Thus, the probability that both will be male is

   $$\begin{aligned} P(A \cap B) &= P(A)P(B \mid A) \\ &= \frac{45}{70} \cdot \frac{44}{69} \\ &= \frac{1980}{4830} \\ &= 41\%. \end{aligned}$$

   (b) What is the probability that both students are female?

   Solution: Using the events A and B defined in the previois part, $P(A^c) = \frac{25}{70}$ and $P(B^c \mid A^c) = \frac{24}{69}$. Thus, the probability that both will be female is

   $$\begin{aligned} P(A^c \cap B^c) &= P(A^c)P(B^c \mid A^c) \\ &= \frac{25}{70} \cdot \frac{24}{69} \\ &= \frac{600}{4830} \\ &= 12\%. \end{aligned}$$

   (c) What is the probability that one of the students is male and one of the students is female?

   Solution: The event "one student is male and the other is female" is equivalent to the compound event $(A \cap B^c) \cup (A^c \cap B)$; that is, either the first is male and the second is female, or the first is female and the second is male. Since $A \cap B^c$ and $A^c \cap B$ are mutually exclusive, it follows that

   $$P(\text{one male and one female}) = P(A \cap B^c) + P(A^c \cap B).$$

Using the multiplicative rule,

$$\begin{aligned}P(A \cap B^c) &= P(A)P(B^c \mid A)\\&= \frac{45}{70}\frac{25}{69}\\&= \frac{1125}{4830}\end{aligned}$$

$$\begin{aligned}P(A^c \cap B) &= P(A^c)P(B \mid A^c)\\&= \frac{25}{70}\frac{45}{69}\\&= \frac{1125}{4830}.\end{aligned}$$

Thus,

$$\begin{aligned}P(\text{one male and one female}) &= \frac{1125}{4830} + \frac{1125}{4830}\\&= \frac{2250}{4830}\\&= 47\%.\end{aligned}$$

5. Of the 64 students who filled out the survey, 13 indicated that they are are currently employed, either at an internship or a paid position ($13/64 = 20\%$). Suppose that we randomly select two different survey respondents.

   (a) What is the probability that both students are currently working?

   Solution:
   $$\frac{13}{64} \cdot \frac{12}{63} = \frac{156}{4032} = 4\%.$$

   (b) What is the probability that neither student is currently working?

   Solution:
   $$\frac{51}{64} \cdot \frac{50}{63} = \frac{2550}{4032} = 63\%.$$

   (c) What is the probability that exactly one student is currently working?

   Solution:
   $$\frac{51}{64} \cdot \frac{13}{63} + \frac{13}{64}\frac{51}{63} = \frac{1326}{4032} = 33\%.$$