Confidence Intervals – Solutions

STAT-UB.0103 – Statistics for Business Control and Regression Models

Population Mean (Known Variance)

1. A random sample of n measurements was selected from a population with unknown mean μ and known standard deviation σ. Calculate a 95% confidence interval for μ for each of the following situations:

   (a) $n = 49$, $\bar{x} = 28$, $\sigma = 28$

   Solution: The formula for the 95% confidence interval is

   $$\bar{x} \pm 2\frac{\sigma}{\sqrt{n}}.$$

   If we substitute the values of the problem, we get $28 \pm 2\frac{28}{\sqrt{49}}$, i.e.

   $$28 \pm 8,$$

   or $(20, 36)$.

   (b) $n = 100$, $\bar{x} = 125$, $\sigma = 50$

   Solution: $125 \pm 2\frac{50}{\sqrt{100}}$, i.e.

   $$125 \pm 10.$$

   (c) Is the assumption that the underlying population of measurements is normally distributed necessary to ensure the validity of the confidence intervals in parts (a)–(c)?

   Solution: No. Since the sample sizes are large ($n \geq 30$), the central limit theorem guarantees that $\bar{x}$ is approximately normal, so the confidence intervals are valid.

2. A random sample of 36 measurements was selected from a population with unknown mean μ and known standard deviation $\sigma = 18$. The sample mean is $\bar{x} = 12$. Calculate a 95% confidence interval for μ.

> **Solution:** We compute a 95% confidence interval for μ via the formula $\bar{x} \pm 2\frac{\sigma}{\sqrt{n}}$. In this case, we get $12 \pm 2\frac{18}{\sqrt{36}}$ i.e., 12 ± 6.

3. With respect to the previous problem, which of the following statements are true:

   A. There is a 95% chance that μ is between 6 and 18.

   B. The population mean μ will be between 6 and 18 about 95% of the time.

   C. In 95% of all future samples, the sample mean will be between 6 and 18.

   D. The population mean μ is between 6 and 18.

   E. None of the above.

> **Solution:** The correct answer is E. The numbers μ, 6, and 18 are all nonrandom, so it makes no sense to talk about probability. Instead, we can say that we have 95% *confidence* that μ is between 6 and 18. The term "confidence" denotes subjective belief, as opposed to "probability," which is concerned with randomness.

4. Complete Problem 2, with a 99% confidence interval instead of a 95% confidence interval.

> **Solution:** For a $100(1-\alpha)\%$ confidence interval for μ, we use the formula $\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$. For a 99% confidence interval, we have $\alpha = .01$ and $z_{\alpha/2} = 2.576$. Thus, our confidence interval for μ is $12 \pm 2.576\frac{18}{\sqrt{36}}$ i.e., 12 ± 7.728.

5. Complete Problem 2, with an 80% confidence interval instead of a 95% confidence interval.

> **Solution:** For a $100(1-\alpha)\%$ confidence interval for μ, we use the formula $\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$. For a 80% confidence interval, we have $\alpha = .20$ and $z_{\alpha/2} = 1.282$. Thus, our confidence interval for μ is $12 \pm 1.282\frac{18}{\sqrt{36}}$ i.e., 12 ± 3.846.

Population Mean (Unknown Variance)

6. A random sample of n measurements was selected from a population with unknown mean μ and unknown standard deviation σ. Calculate a 95% confidence interval for μ for each of the following situations:

   (a) $n = 25$, $\bar{x} = 28$, $s = 12$

   Solution: When we don't know the population standard deviation, we use a t-based confidence interval with $n-1$ degrees of freedom. The formula for the 95% confidence interval is

   $$\bar{x} \pm t_{.025} \frac{s}{\sqrt{n}},$$

   where $t_{.025}$ is the critical value for a t random variable with $n-1$ degrees of freedom. Since $n = 25$, we need the critical value for $n - 1 = 24$ degrees of freedom. Using the t table, this value is $t_{.025} = 2.064$ The 95% confidence interval is $28 \pm 2.064 \cdot \frac{12}{\sqrt{25}}$, i.e.

   $$28 \pm 4.9536.$$

   (b) $n = 16$, $\bar{x} = 12$, $s = 18$

   Solution: We use $n - 1 = 15$ degrees of freedom. Using the t table, $t_{.025} = 2.131$. The 95% confidence interval is $12 \pm 2.131 \cdot \frac{18}{\sqrt{16}}$, i.e.

   $$12 \pm 9.5895.$$

   (c) $n = 100$, $\bar{x} = 125$, $s = 50$

   Solution: We use $n - 1 = 99$ degrees of freedom. The closest value in the table is 60 degrees of freedom. We use $t_{.025} \approx 2.000$. The 95% confidence interval is $125 \pm 2.000 \cdot \frac{50}{\sqrt{100}}$, i.e.

   $$125 \pm 10.$$

   (d) Is the assumption that the underlying population of measurements is normally distributed necessary to ensure the validity of the confidence intervals in parts (a)–(c)?

   Solution: We need this assumption for (a) and (b), but not for (c). In (c), the sample size is above 30, so the central limit theorem applies, and the t is a reasonable approximation.

7. In each of the following situations, find α and $t_{\alpha/2}$.

   (a) An 80% confidence interval with $n = 10$.

   > **Solution:**
   >
   > $$\alpha = .20, \quad n-1 = 9 \text{ degrees of freedom}, \quad t_{.100} = 1.383.$$

   (b) A 99% confidence interval with $n = 25$.

   > **Solution:**
   >
   > $$\alpha = .01, \quad n-1 = 24 \text{ degrees of freedom}, \quad t_{.005} = 2.797$$

   (c) A 90% confidence interval with $n = 30$.

   > **Solution:**
   >
   > $$\alpha = .10, \quad n-1 = 29 \text{ degrees of freedom}, \quad t_{.050} = 1.699$$

8. Compute $z_{\alpha/2}$ for each of the situations in problem 7.

> **Solution:** We can look up the answer in the ∞ degrees of freedom column. The answers are: (a) $z_{.100} = 1.282$; (b) $z_{.005} = 2.576$; (c) $z_{.050} = 1.645$.

9. The white wood material used for the roof of an ancient Japanese temple is imported from Northern Europe. The wooden roof must withstand as much as 100 centimeters of snow in the winter. Architects at Tohoku University (Japan) conducted a study to estimate the mean bending strength of the white wood roof. A sample of 25 pieces of the imported wood were tested and yielded the following statistics on breaking strength: $\bar{x} = 74.5$, $s = 10.9$. Estimate the true mean breaking strength of the white wood with a 90% confidence interval.

> **Solution:** We don't know the population standard deviation so we use a t-based confidence interval. The formula for the interval is $\bar{x} \pm t_{\alpha/2}\frac{s}{\sqrt{n}}$. We have $\alpha = .10$ and $n-1 = 24$ degrees of freedom, so $t_{\alpha/2} = t_{.05} = 1.711$. The 90% confidence interval is $74.5 \pm 1.711 \cdot \frac{10.9}{\sqrt{25}}$, i.e.
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> $$74.5 \pm 3.723.$$

10. Researchers recorded expenses per full-time equivalent employee for each in a sample of 1,751 army hospitals. The sample yielded the following summary statistics: $\bar{x} = \$6,563$ and $s = \$2,484$. Estimate the mean expenses per full-time equivalent employee of all U.S. army hospitals using a 90% confidence interval.

Solution: Again, we use a t-based confidence interval. Now, we have $\alpha/2 = .10/2 = .05$ and $n-1 = 1750$ degrees of freedom. We use the largest value of degrees of freedom in the table (120), so $t_{\alpha/2} \approx 1.658$. The 90% confidence interval is $6563 \pm 1.658 \cdot \frac{2484}{\sqrt{1751}}$, i.e.

$$6563 \pm 98.42.$$