Expectation and Variance – Solutions
STAT-UB.0103 – Statistics for Business Control and Regression Models

Random variables (review)

1. Let X be a random variable describing the number of cups of coffee a randomly-chosen NYU undergraduate drinks in a week. Suppose that there is a 10% chance that the student has one cup of coffee, 30% chance that the student has two cups of coffee, 40% chance that the student has 3 cups of coffee, and a 20% chance stat the student has four cups of coffee.

   (a) Let $p(x)$ be the probability distribution function of X. Fill in the following table:

x	1	2	3	4
$p(x)$				

Solution:

x	1	2	3	4
$p(x)$	.10	.30	.40	.20

   (b) Find $\mathrm{E}(X)$, the expectation of X.

Solution:

$$\begin{aligned}\mathrm{E}(X) &= (.10)(1) + (.30)(2) + (.40)(3) + (.20)(4)\\ &= 2.7.\end{aligned}$$

   (c) What is the interpretation of the expectation of X?

Solution: The long-run sample mean. If we performed the random experiment upon which X is measured many times, getting a different value of X each time, then the sample mean would be very close to the expectation of X.

Variance and Standard Deviation

2. This is a continuation of problem 1.

(a) Find $\text{var}(X)$ and $\text{sd}(X)$, the variance and standard deviation of X.

Solution:

$$\begin{aligned}\text{var}(X) &= (.10)(1-2.7)^2 + (.30)(2-2.7)^2 + (.40)(3-2.7)^2 + (.20)(4-2.7)^2 \\ &= .81.\end{aligned}$$

The standard deviation of X is given by

$$\text{sd}(X) = \sqrt{\text{var}(X)} = 0.9.$$

(b) What is the interpretation of the standard deviation of X?

Solution: The long-run sample standard devation. If we performed the random experiment upon which X is measured many times, getting a different value of X each time, then the sample standard deviation would be very close to the standard deviation of X. If the PDF of X is symmetric and mound-shaped, we can use the empirical rule to make predictions about the value of X.

3. Consider the following game:

    1. You pay \$6 to pick a card from a standard 52-card deck.
    2. If the card is a diamond ($\diamondsuit$), you get \$22; if the card is a heart ($\heartsuit$), you get \$6; otherwise, you get nothing.

   Perform the following calculations to decide whether or not you would play this game.

   (a) Let W be the random variable equal to the amount of money you win from playing the game. If you lose money, W will be negative. Find the PDF of W.

   Solution: The sample points corresponding to the suit of the card are $\spadesuit$, $\heartsuit$, $\clubsuit$, and $\diamondsuit$; each of these has probability $\frac{1}{4}$. The values of the random variable W corresponding to the sample points are as follow:

Outcome	W
$\spadesuit$	-6
$\heartsuit$	0
$\clubsuit$	-6
$\diamondsuit$	16

   Thus, the PDF of W is given by the table:

w	-6	0	16
$p(w)$	0.50	0.25	0.25

   (b) What are your expected winnings? That is, what is μ, the expectation of W?

   Solution:
   Using the PDF computed in part (a), the expected value of W is

   $$\begin{aligned}\mu &= (.50)(-6) + (.25)(0) + (.25)(16) \\ &= 1.\end{aligned}$$

   On average, we win \$1 every time we play the game.

(c) What is the standard deviation of W?

Solution:
Using the PDF computed in part (a), and the expected value computed in part (b), we compute the variance of W as

$$\begin{aligned}\sigma^2 &= (.50)(-6-1)^2 + (.25)(0-1)^2 + (.25)(16-1)^2 \\ &= 81.\end{aligned}$$

Thus, the standard deviation of W is

$$\sigma = \sqrt{81} = 9.$$

(d) What are the interpretations of the expectation and standard deviation of W?

Solution: If we played the game many many times, then the average of our winnings over all times we played would be close to the \$1, and the standard deviations of our winnings over all times we played would be close to \$9.

Properties of Expectation and Variance

4. **Affine Transformations.** Let X be a random variable with expectation $\mu_X = 2$ and standard deviation $\sigma_X = 3$.

   (a) What is the expectation of $5X + 2$?

   Solution:
   $$5\mu_X + 2 = 12.$$

   (b) What is the standard deviation of $5X + 2$?

   Solution:
   $$|5|\sigma_X = 15.$$

5. **Sums of Independent Random Variables.** Let X and Y be independent random variables with $\mu_X = 1$, $\sigma_X = 3$, $\mu_Y = -5$, $\sigma_Y = 4$.

   (a) What is $\mathrm{E}(X + Y)$?

   Solution:
   $$\mathrm{E}(X + Y) = \mu_X + \mu_Y = 1 + (-5) = -4.$$

   (b) Find $\mathrm{var}(X + Y)$ and $\mathrm{sd}(X + Y)$.

   Solution:
   $$\begin{aligned} \mathrm{var}(X + Y) &= \sigma_X^2 + \sigma_Y^2 = (3)^2 + (4)^2 = 25, \\ \mathrm{sd}(X + Y) &= \sqrt{\mathrm{var}(X + Y)} = 5. \end{aligned}$$

6. Let X and Y be independent random variables with $\mu_X = -2$, $\sigma_X = 1$, $\mu_Y = 3$, $\sigma_Y = 4$.

   (a) Find the expectation and standard deviation of $-3X + 2$.

   Solution:
   $$\begin{aligned} \mathrm{E}(-3X + 2) &= -3\mu_X + 2 = -3(-2) + 2 = 8, \\ \mathrm{sd}(-3X + 2) &= |-3|\sigma_X = 3(1) = 3. \end{aligned}$$

(b) Find the expectation and standard deviation of $X + Y$.

Solution:

$$\begin{aligned} \mathrm{E}(X+Y) &= \mu_X + \mu_Y = 1, \\ \mathrm{var}(X+Y) &= \sigma_X^2 + \sigma_Y^2 = (1)^2 + (4)^2 = 17, \\ \mathrm{sd}(X+Y) &= \sqrt{\mathrm{var}(X+Y)} = \sqrt{17}. \end{aligned}$$

(c) Find the expectation and standard deviation of $-3X + Y + 2$.

Solution:

$$\begin{aligned} \mathrm{E}(-3X+Y+2) &= -3\mu_X + \mu_Y + 2 = 11, \\ \mathrm{var}(-3X+Y+2) &= (-3)^2\sigma_X^2 + \sigma_Y^2 = (-3)^2(1)^2 + (4)^2 = 25, \\ \mathrm{sd}(-3X+Y+2) &= \sqrt{\mathrm{var}(-3X+Y+2)} = 5. \end{aligned}$$

Advanced Problems

7. **Bernoulli random variable.** Suppose you flip a biased coin that lands Heads with probability p and lands tails with probability $1 - p$. Define the random variable

$$X = \begin{cases} 1 & \text{if the coin lands Heads;} \\ 0 & \text{if the coin lands Tails.} \end{cases}$$

This random variable is called a "Bernoulli random variable with success probability p."

(a) What is the PDF of X?

Solution:

x	0	1
$p(x)$	$1-p$	p

(b) Find μ, the expectation of X

Solution:

$$\mu = (1-p)(0) + (p)(1) = p.$$

(c) Find σ^2, the variance of X.

Solution:

$$\sigma^2 = (1-p)(0-p)^2 + (p)(1-p)^2 = p(1-p).$$

8. Suppose you have a biased coin that lands Heads with probability p and lands Tails with probability $1 - p$. You flip the coin 2 times. Let Y be the number of times the coin lands Heads.

(a) What is $\mathrm{E}(Y)$?

Solution:

$$\mathrm{E}(Y) = p + p = 2p.$$

(b) What is $\mathrm{var}(Y)$?
Hint: $Y = X_1 + X_2$, where X_1 and X_2 are independent Bernoulli random variables corresponding to the 2 coin flips. Use the answer to problem 7(c).

Solution:

$$\mathrm{var}(Y) = p(1-p) + p(1-p).$$

(c) Suppose instead that you flip the coin n times, and let Y count the number of Heads. What are the expectation and variance of Y?
Hint: $Y = X_1 + X_2 + \cdots + X_n$.

Solution:

$$\mathrm{E}(Y) = np; \quad \mathrm{var}(Y) = np(1-p).$$