One-sided alternatives

1. The average nicotine content of a brand of cigarettes must be less than 0.5 mg for it to qualify as a Low Nicotine brand. The manufacturer of Lucky Strikes Cigarettes claims that it is a Low Nicotine brand. To test this claim, the FDA takes a random sample of 20 cigarettes (one pack) of Lucky Strikes. They find an average nicotine content of 0.4 mg, with a sample standard deviation of 0.2 mg. Test the manufacturer's claim, at the 1% level of significance. Assume that the nicotine measurements are normally distributed.

   (a) What are the population and the sample?

   > **Solution:** The population is the nicotine contents of all Low Nicotine cigarettes man-ufactured by Lucky Strike. The sample is the $n = 20$ sampled cigarettes.

   (b) What are the null and alternative hypotheses?

   > **Solution:** Let μ be the average nicotine contents (in mg) of all cigarettes in the population.
   >
   > $$H_0 : \mu = 0.5$$
   > $$H_a : \mu > 0.5$$

   (c) What is the test statistic?

   > **Solution:** Since the population standard devation is unknown, we use a t statistic:
   >
   > $$T = \frac{\bar{X} - 0.5}{S/\sqrt{n}}$$

   (d) What is the rejection region?

   > **Solution:** We reject H_0 if $T > t_\alpha$ for $\alpha = 0.01$ and $n - 1 = 19$ degrees of freedom. That is, we reject H_0 if $T > 2.539$.

   (e) What assumptions are you making?

   > **Solution:** We assume that the sample is unbiased. Also, since $n < 30$, we need to assume that the population is normally distributed.

   (f) What is the result of the test?

   > **Solution:** The observed test statistic is
   >
   > $$t = \frac{0.4 - 0.5}{0.2/\sqrt{20}} = -2.236.$$

Since this is not in the rejection region, we do not reject the null hypothesis.

2. The manager of a credit card company claims that the mean time to settle disputed charges is 30 days. A regulator is worried that the manager's claim is too optimistic. The regulator examines a random sample of 15 disputed charges, and finds a mean time to settlement of 35.9 days, with a sample standard deviation of 10.2 days. Is there evidence at the 5% level of significance to doubt the manager's claim, assuming that the time to settle disputes is normally distributed?

(a) What are the population and the sample?

> **Solution:** The population consists of the times to settle all disputed charges handled by the credit card company. The sample the nicotine contents of all Low Nicotine cigarettes manufactured by Lucky Strike. The sample is the $n = 30$ sampled claims.

(b) What are the null and alternative hypotheses?

> **Solution:** Let μ be the average nicotine contents (in mg) of all cigarettes in the population.
>
> $$H_0 : \mu = 30$$
> $$H_a : \mu > 30$$

(c) What is the test statistic?

> **Solution:** Since the population standard devation is unknown, we use a t statistic:
>
> $$T = \frac{\bar{X} - 30}{S/\sqrt{n}}$$

(d) What is the rejection region?

> **Solution:** We reject H_0 if $T > t_\alpha$ for $\alpha = 0.05$ and $n - 1 = 14$ degrees of freedom. That is, we reject H_0 if $T > 1.761$.

(e) What assumptions are you making?

> **Solution:** We assume that the sample is unbiased. Also, since $n < 30$, we need to assume that the population is normally distributed.

(f) What is the result of the test?

> **Solution:** The observed test statistic is
>
> $$t = \frac{35.9 - 30}{10.2/\sqrt{15}} = 2.240.$$
>
> Since this is in the rejection region, we reject the null hypothesis.

Observed significance levels (p-values)

3. Suppose you want to perform a hypothesis test with a two-sided alternative ($H_0 : \mu = \mu_0$, $H_a : \mu \neq \mu_0$) using a z statistic.

   (a) If the observed test statistic is $z = 1.8$, would a level 5% test reject H_0? Would a level 10% reject H_0?
   Note: $z_{.100} = 1.282$, $z_{.050} = 1.645$, $z_{.025} = 1.960$

   > **Solution:** For a level 5% test, we reject H_0 if $|z| > z_{.025} = 1.960$. For a level 10% test, we reject H_0 if $|z| > z_{.050} = 1.645$. Thus, a level 10% test would reject, but a level 5% test would not.

   (b) What values of α would a level α test reject H_0 if $z = 1.8$?
   Note: $\Phi(1.8) = .9641$

   > **Solution:** We reject H_0 if $|z| > z_{\alpha/2}$, i.e. $1.8 > z_{\alpha/2}$. This will happen if $\alpha/2 > (1 - \Phi(1.8))$:
   >
   > $$\alpha/2 > 1 - .9641$$
   > $$\alpha > .0718$$
   >
   > That is, a level α test will reject H_0 if $\alpha > .0718$.

   (c) For this hypothesis test, what is the p-value corresponding to $z = 1.8$? That is, if H_0 is true and we repeat the random experiment, what is the chance of getting a test statistic at least as extreme as the value $z = 1.8$?

   > **Solution:** We computed this in the previous problem: $p = .0718$.

4. Suppose you want to perform a hypothesis test with a one-sided, greater-than alternative ($H_0 : \mu = \mu_0$, $H_a : \mu > \mu_0$) using a z statistic.

   (a) If the observed test statistic is $z = 1.8$, would a level 5% test reject H_0? Would a level 10% reject H_0?
   Note: $z_{.100} = 1.282$, $z_{.050} = 1.645$, $z_{.025} = 1.960$

   > **Solution:** For a level 5% test, we reject H_0 if $|z| > z_{.050} = 1.645$. For a level 10% test, we reject H_0 if $|z| > z_{.100} = 1.282$. Thus, both a level 10% and a level 5\$ test would reject H_0.

(b) What values of α would a level α test reject H_0 if $z = 1.8$?
Note: $\Phi(1.8) = .9641$

> **Solution:** We reject H_0 if $|z| > z_\alpha$, i.e. $1.8 > z_\alpha$. This will happen if $\alpha > (1 - \Phi(1.8))$:
>
> $$\alpha > 1 - .9641$$
> $$\alpha > .0359$$
>
> That is, a level α test will reject H_0 if $\alpha > .0359$.

(c) If H_0 is true and we repeat the random experiment, what is the chance of getting a test statistic at least as extreme as the value $z = 1.8$?

> **Solution:** We computed this in the previous problem: .0359.

5. In the "Quarter Pounder" example, we we tested the null hypothesis that the weight of a McDonald's quarter pounder is 0.25 pounds ($H_0 : \mu = 0.25$) against the alternative that the weight is below 0.25 pounds ($H_a : \mu < 0.25$). After collecting a sample our observed z statistic was $z = -2.02$. Find the largest level α at which the hypothesis testing procedure does not reject H_0.

> **Solution:** This problem has hypotheses
>
> $$H_0 : \mu = 0.25,$$
> $$H_a : \mu < 0.25.$$
>
> We reject H_0 for small values of the Z statistic, i.e. we reject when $Z < -z_\alpha$. The smallest level α at which we could perform a test and still reject H_0 is the value at which $z = -z_\alpha$. Since $\alpha = \Phi(-z_\alpha)$, we have
> $$\alpha = \Phi(z) = 2.17\%.$$

6. Suppose we perform a hypothesis test and we observe a p-value of $p = .02$. True or false: There is a 2% chance that the null hypothesis is true.

> **Solution:** False. The p-value is the probability of getting a test statistic at least as extreme as what was observed. Heuristically, we can think of this as
>
> $$\mathrm{P}(\text{Data} \mid H_0 \text{ is true}) = 2\%.$$
>
> The statement in the problem is
>
> $$\mathrm{P}(H_0 \text{ is true} \mid \text{Data}) = 2\%.$$
>
> Clearly, this is not the same.

7. Suppose we perform a hypothesis test and we observe a p-value of $p = .02$. True or false: If we reject the null hypothesis, then there is a 2% chance of making a type I error.

> **Solution:** False. We can only make a type I error when the null hypothesis is true. Thus, the statement in question 7 is *exactly the same* as the statement in question 6.

8. Suppose we perform a hypothesis test and we observe a Z test statistic $z = -2.02$, corresponding to a p-value of $p = .02$. True or false: If we were to repeat the experiment and the null hypothesis were actually true, then there would be a 2% chance of observing a test statistic at least as extreme as $z = -2.02$.

Solution: True. The p-value is the probability of getting a test statistic as least as extreme as the observed value if the null hypothesis were true. Note: for a one-sided less-than alternative, extreme means "less than or equal to."