Normal Random Variables – Solutions

STAT-UB.0103 – Statistics for Business Control and Regression Models

Standard normal random variables

1. Suppose Z is a standard normal random variable. What is $P(Z \leq 1.2)$?

Solution:

$$P(Z \leq 1.2) = \Phi(1.2) = .8849.$$

We have computed $\Phi(z)$ by using a normal table.

2. Suppose Z is a standard normal random variable. What is $P(Z \leq -2.36)$?

Solution:

$$P(Z \leq -2.36) = \Phi(-2.36) = .0091.$$

We have computed $\Phi(z)$ by using a normal table.

3. Suppose Z is a standard normal random variable. What is $P(Z \leq -0.41)$

Solution:

$$P(Z \leq -0.41) = \Phi(-0.41) = .3409.$$

4. Suppose Z is a standard normal random variable. What is $P(-0.41 \leq Z \leq 1.2)$?

Solution:

$$\begin{aligned} P(-0.41 \leq Z \leq 1.2) &= P(Z \leq 1.2) - P(Z \leq -0.41) \\ &= \Phi(1.2) - \Phi(-0.41) \\ &= .8849 - .3409 \\ &= .5440. \end{aligned}$$

5. Suppose Z is a standard normal random variable. What is $P(Z > 1.96)$?

Solution:

$$\begin{aligned} P(Z > 1.96) &= 1 - P(Z \leq 1.96) \\ &= 1 - \Phi(1.96) \\ &= 1 - .9750 \\ &= .0250. \end{aligned}$$

Normal CDF

6. The dressed weights of Excelsior Chickens are approximately normally distributed with mean 3.20 pounds and standard deviation 0.40 pound. About what proportion of the chickens have dressed weights greater than 3.60 pounds?

Solution: Let X be the weight of a typical chicken in pounds; this is normally distributed with mean $\mu = 3.20$ and standard deviation $\sigma = 0.40$. The proportion of chickens with dressed weights greater than 3.60 is equal to the probability that X is greater than 3.60 pounds.

Define $Z = (X - \mu)/\sigma$, a standard normal random variable. Then,

$$\begin{aligned}
\mathrm{P}(X > 3.60) &= \mathrm{P}\left(\frac{X-\mu}{\sigma} > \frac{3.60-\mu}{\sigma}\right) \\
&= \mathrm{P}\left(Z > \frac{3.60-3.20}{0.40}\right) \\
&= \mathrm{P}(Z > 1) \\
&= 1 - \mathrm{P}(Z \leq 1) \\
&= 1 - \Phi(1) \\
&= 1 - .8413 \\
&= .1587.
\end{aligned}$$

7. Suppose that an automobile muffler is designed so that its lifetime (in months) is approximately normally distributed with mean 26 months and standard deviation 4 months.

(a) The manufacturer has decided to use a marketing strategy in which the muffler is covered by warranty for 18 months. Approximately what proportion of the mufflers will fail before the warranty expires?

Solution: Let X be the lifetime of a typical muffler in months; this is normally distributed with mean $\mu = 26$ and standard deviation $\sigma = 4$. The muffler will fail before the warranty expires if and only if $X < 18$. Thus, the proportion of the mufflers that fail before the warranty expires is equal to the probability that X is less than 18. Define $Z = (X - \mu)/\sigma$, a standard normal random variable. Then,

$$\begin{aligned} \mathrm{P}(X < 18) &= P\Big(\frac{X-\mu}{\sigma} < \frac{18-26}{4}\Big) \\ &= P(Z < -2) \\ &= \Phi(-2) \\ &= .0228. \end{aligned}$$

(b) Suppose that the manufacturer in the previous example would like to extend the warranty time to 24 months. Approximately what proportion of the mufflers will fail before the extended warranty expires?

Solution: This is similar to part (a), but now we need to compute $P(X < 24)$:

$$\begin{aligned} \mathrm{P}(X < 24) &= P\Big(\frac{X-\mu}{\sigma} < \frac{24-26}{4}\Big) \\ &= P(Z < -0.5) \\ &= \Phi(-0.5) \\ &= .3085. \end{aligned}$$

(c) Of all the mufflers that fail under the extended warranty, what proportion of them have failures in the interval (18 months, 24 months)?

Solution: The requested quantity is equal to the probability that X is in the interval $(18, 24)$, *conditional that X is less than* 24. Using the definition of conditional probability, we compute

$$
\begin{aligned}
\mathrm{P}(18 < X < 24 \mid X < 24) &= \frac{\mathrm{P}(18 < X < 24 \text{ and } X < 24)}{\mathrm{P}(X < 24)} \\
&= \frac{\mathrm{P}(18 < X < 24)}{\mathrm{P}(X < 24)} \\
&= \frac{\mathrm{P}(X < 24) - \mathrm{P}(X < 18)}{\mathrm{P}(X < 24)} \\
&= \frac{.3085 - .0228}{.3085} \\
&= .9261.
\end{aligned}
$$

Inverse Normal CDF

8. Suppose that Z is a standard normal random variable. Find the value w so that $\mathrm{P}(|Z| \leq w) = 0.60$.

Solution: The problem is asking for w such that $\mathrm{P}(-w \leq Z \leq w) = 0.60$. Note that

$$\mathrm{P}(Z < -w) + \mathrm{P}(-w \leq Z \leq w) + P(Z > w) = 1.$$

Also, $\mathrm{P}(Z < -w) = \mathrm{P}(Z > w)$ (draw a picture if this is not obvious to you). In this case, we must have that $\mathrm{P}(Z < -w) = 0.20$. Therefore, $\mathrm{P}(Z < w) = 0.80$.

Now, $\Phi(w) = 0.80$. The normal CDF table tells us that $\Phi(0.84) = .7795$ and $\Phi(0.85) = .8023$; we take w to be the closer of these two values, i.e. $w \approx 0.84$.

9. A machine that dispenses corn flakes into packages provides amounts that are approximately normally distributed with mean weight 20 ounces and standard deviation 0.6 ounce. Suppose that the weights and measures law under which you must operate allows you to have only 5% of your packages under the weight stated on the package. What weight should you print on the package?

Solution: Let X be the weight in ounces of a typical package; this is approximately normally distributed with mean $\mu = 20$ and standard deviation $\sigma = 0.6$. We seek a printed weight, w, such that $P(X < w) = .05$. Define $Z = (X - \mu)/\sigma$, a standard normal random variable. We have the following relation:

$$.05 = \mathrm{P}(X < w) = \mathrm{P}\left(\frac{X - \mu}{\sigma} < \frac{w - 20}{0.6}\right) = \mathrm{P}\left(Z < \frac{w - 20}{0.6}\right) = \Phi\left(\frac{w - 20}{0.6}\right).$$

Thus,

$$\frac{w - 20}{0.6} = \Phi^{-1}(.05).$$

With a normal table, we compute $\Phi(-1.64) = .0505$ and $\Phi(-1.65) = .0495$, so $\Phi^{-1}(.05) \approx -1.645$. Finally,

$$\frac{w - 20}{0.6} = 1.645,$$

so $w = 20 + 0.6 \times 1.645 = 19.01$. We would probably round this to 19.01 and print "19 ounces" on the box.

More examples

10. Suppose that the daily demand for change (meaning coins) in a particular store is approximately normally distributed with mean \$800.00 and standard deviation \$60.00.

(a) What is the probability that, on any particular day, the demand for change will be below \$600?

Solution: Let X be the demain for change on a particular day (in dollars); this is a normal random variable with mean $\mu = 800$ and standard deviation $\sigma = 60$. Now

$$\mathrm{P}(X < 600) = \mathrm{P}\left(\frac{X - \mu}{\sigma} < \frac{600 - 800}{60}\right) = \Phi(-3.33) = .0004.$$

(b) Find the amount M of change to keep on hand if one wishes, with certainty 99%, to have enough change. That is, find M so that $\mathrm{P}(X \leq M) = 0.99$.

Solution: We have

$$.99 = P(X \leq M) = P\left(\frac{X-\mu}{\sigma} \leq \frac{M-800}{60}\right) = \Phi\left(\frac{M-800}{60}\right).$$

Thus,

$$\frac{M-800}{60} = \Phi^{-1}(.99).$$

Consulting the normal CDF table, we see that $\Phi(2.32) = .9898$ and $\Phi(2.33) = .9901$. We take $\Phi^{-1}(.99) \approx 2.33$, so that

$$M = 800 + 60 \times 2.33 = 939.80.$$