

Binomial random variables (Review)

1. Suppose that you are rolling a die eight times. Find the probability that the face with two spots comes up exactly twice.

Solution: Let X be the number of times that we get the face with two spots. This is a binomial random variable with $n = 8$ and $p = \frac{1}{6}$. We compute

$$\begin{aligned} P(X = 2) &= \binom{8}{2} p^2 (1-p)^{8-2} \\ &= \binom{8}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^6 \\ &\approx 0.26. \end{aligned}$$

2. A new restaurant opening in Greenwich village has a 30% chance of survival during their first year. If 16 restaurants open this year, find the probability that exactly 3 restaurants survive.

Solution: Let X be the number that survive. This is a binomial random variable with $n = 16$ and $p = 0.3$. Therefore,

$$\begin{aligned} P(X = 3) &= \binom{16}{3} (0.3)^3 (1 - 0.3)^{(16 - 3)} \\ &= .146 \end{aligned}$$

3. The probability of winning at a certain game is 0.10. If you play the game 10 times, what is the probability that you win at most once?

Solution: Let X be the number of times that we win. This is a binomial random variable with $n = 10$ and $p = 0.10$. We compute

$$\begin{aligned} P(X \leq 1) &= P(X = 0) + P(X = 1) \\ &= \binom{n}{0} p^0 (1-p)^{n-0} + \binom{n}{1} p^1 (1-p)^{n-1} \\ &= \binom{10}{0} (0.10)^0 (0.90)^{10} + \binom{10}{1} (0.10)^1 (0.90)^9 \\ &= (0.90)^{10} + 10(0.10)(0.90)^9 \\ &\approx 0.736. \end{aligned}$$

4. The probability is 0.3 that an audit of a retail business will turn up irregularities in the collection of state sales tax. If 16 retail businesses are audited, find the probability that
- (a) fewer than 5 will have irregularities in the collection of state sales tax.

Solution: Let X be the number audited. This is a binomial random variable with $n = 16$ and $p = 0.3$. Therefore,

$$\begin{aligned} P(X < 5) &= \binom{16}{0} (0.3)^0 (0.7)^{16} + \binom{16}{1} (0.3)^1 (0.7)^{15} + \binom{16}{2} (0.3)^2 (0.7)^{14} \\ &\quad + \binom{16}{3} (0.3)^3 (0.7)^{13} + \binom{16}{4} (0.3)^4 (0.7)^{12} \\ &\approx .450. \end{aligned}$$

- (b) more than 5 will have irregularities in the collection of state sales tax.

Solution:

$$\begin{aligned} P(X > 5) &= 1 - P(X \leq 5) \\ &= 1 - \left[\binom{16}{0} (0.3)^0 (0.7)^{16} + \binom{16}{1} (0.3)^1 (0.7)^{15} + \binom{16}{2} (0.3)^2 (0.7)^{14} \right. \\ &\quad \left. + \binom{16}{3} (0.3)^3 (0.7)^{13} + \binom{16}{4} (0.3)^4 (0.7)^{12} + \binom{16}{5} (0.3)^5 (0.7)^{11} \right] \\ &\approx .340. \end{aligned}$$

Poisson random variables

5. The number of calls arriving at the Swampside Police Station follows a Poisson distribution with rate 4.6/hour. What is the probability that exactly six calls will come between 8:00 p.m. and 9:00 p.m.?

Solution: Let X be the number of calls that arrive between 8:00 p.m. and 9:00 p.m. This is a Poisson random variable with mean

$$\lambda = E(X) = (4.6 \text{ calls/hour})(1 \text{ hour}) = 4.6 \text{ calls.}$$

Thus,

$$P(X = 6) = \frac{\lambda^6}{6!} e^{-\lambda} = \frac{(4.6)^6}{6!} e^{-4.6}.$$

6. In the station from Problem 5, find the probability that exactly 7 calls will come between 9:00 p.m. and 10:30 p.m.

Solution: Let X be the number of calls that arrive between 9:00 p.m. and 10:30 p.m. This is a Poisson random variable with mean

$$\lambda = E(X) = (4.6 \text{ calls/hour})(1.5 \text{ hours}) = 6.9 \text{ calls.}$$

Thus,

$$P(X = 7) = \frac{\lambda^7}{7!} e^{-\lambda} = \frac{(6.9)^7}{7!} e^{-6.9}.$$

7. Car accidents occur at a particular intersection in the city at a rate of about 2/year. Estimate the probability of no accidents occurring in a 6-month period.

Solution: Let X be the number of car accidents. This is Poisson random variable with mean

$$\lambda = E(X) = (2 \text{ accidents/year})(0.5 \text{ years}) = 1 \text{ accident.}$$

Thus,

$$P(X = 0) = \frac{\lambda^0}{0!} e^{-\lambda} = e^{-1} \approx .368.$$

8. In the intersection from Problem 7, estimate the probability of two or more accidents occurring in a year.

Solution: Let X be the number of car accidents. This is Poisson random variable with mean

$$\lambda = E(X) = (2 \text{ accidents/year})(1.0 \text{ years}) = 2 \text{ accident.}$$

Thus,

$$\begin{aligned} P(X \geq 2) &= 1 - P(X < 2) \\ &= 1 - \left[\frac{\lambda^0}{0!} e^{-\lambda} + \frac{\lambda^1}{1!} e^{-\lambda} \right] \\ &\approx .594. \end{aligned}$$

Empirical rule with Binomial and Poisson random variables

9. If you flip a fair coin 100 times, would it be unusual to get 42 heads and 58 tails?

Solution: Let X be the number of heads. Then, X is binomial with $n = 100$ and $p = 0.5$. Thus, its expectation and standard deviation are

$$\mu = np = (100)(0.5) = 50,$$

and

$$\sigma = \sqrt{np(1-p)} = \sqrt{(100)(0.5)(1-0.5)} = 5.$$

Since $np \geq 15$ and $n(1-p) \geq 15$, we can use the empirical rule to approximate the distribution of X . Thus, approximately 95% of the time, X will be in the range $\mu \pm 2\sigma$, or (40, 60). So, it would not be unusual to observe $X = 42$.

10. If X is a Poisson random variable with $\lambda = 225$, would it be unusual to get a value of X which is less than 190?

Solution: Set

$$\mu = E(X) = \lambda = 225,$$

$$\sigma = \text{sd}(X) = \sqrt{\lambda} = 15.$$

Define z to be the number of standard deviations above the mean that 190 is, i.e.

$$190 = \mu + \sigma z.$$

Then,

$$z = \frac{190 - \mu}{\sigma} = \frac{-35}{15} \approx -2.33.$$

A value of X which is below 190 is more than 2.33 standard deviations below the mean of X . The empirical rule tells us that observations more than 2 standard deviations away from the mean are unusual (they occur less than 95% of the time). Therefore, values of X below 190 are unusual.

11. The probability is 0.10 that a person reached on a “cold call” by a telemarketer will make a purchase. If the telemarketer calls 200 people, would it be unusual for them to get 30 purchases?

Solution: Let X be the number of purchases. This is a Binomial random variable with size $n = 200$ and success probability $p = 0.10$. Thus, the expectation and standard deviation of X are

$$\mu = np = (200)(.10) = 20$$

$$\sigma = \sqrt{np(1-p)} = \sqrt{(200)(.10)(.90)} = \sqrt{18} \approx 4.2$$

Since $np \geq 15$ and $np(1-p) \geq 15$, the distribution of X can be approximated by the empirical rule. Using the empirical rule approximation, 95% of the time, X will be in the range $\mu \pm 2\sigma$, or (11.6, 28.4), and 99.7% of the time, X will be in the range $\mu \pm 3\sigma$, or (7.4, 32.6). We would see $X \geq 30$ less than 5% of the time. It would be unusual to see $X = 30$, but not highly unusual.

Hypergeometric random variables

12. If we draw 5 cards from a 52-card deck, what is the probability of getting exactly 2 aces?

Solution: Let X be the number of aces in our hand. This is a hypergeometric random variable with $n = 5$ elements in the sample, $N = 52$ elements in the population, and $r = 4$ success elements in the population (namely, $A\spadesuit$, $A\heartsuit$, $A\clubsuit$, and $A\diamondsuit$). Thus,

$$P(X = 2) = \frac{\binom{r}{2} \binom{N-r}{n-2}}{\binom{N}{n}} = \frac{\binom{4}{2} \binom{48}{3}}{\binom{52}{5}} \approx .040.$$

13. Suppose that a shipment of 100 fruit crates has 11 crates in which the fruit shows signs of spoilage. A quality control inspection selects 8 crates at random, opens these selected crates, and then counts the number (out of 8) in which the fruit shows signs of spoilage. What is the probability that exactly two crates in the sample show signs of spoilage?

Solution: Let X be the number of spoiled crates in the sample. This is a hypergeometric random variable with $n = 8$ elements in the sample, $N = 100$ elements in the population, and $r = 11$ success elements in the population (namely, the spoiled crates). Thus,

$$P(X = 2) = \frac{\binom{r}{2} \binom{N-r}{n-2}}{\binom{N}{n}} = \frac{\binom{11}{2} \binom{89}{6}}{\binom{100}{8}} \approx .171.$$

14. There are 70 students in enrolled in a course: 40 freshman, 20 sophomores, and 10 juniors. If you randomly select a group of 4 students, what is the probability of getting all freshman?

Solution: Let X be the number of freshmen in the group. This is a hypergeometric random variable with $n = 4$ elements in the sample, $N = 70$ elements in the population, and $r = 40$ success elements in the population. Therefore,

$$P(X = 4) = \frac{\binom{r}{4} \binom{N-r}{n-4}}{\binom{N}{n}} = \frac{\binom{40}{4} \binom{30}{0}}{\binom{70}{4}} \approx .010.$$