Models for Counts – Solutions
COR1-GB.1305 – Statistics and Data Analysis

Properties of Expectation

1. **Affine Transformations.** Let X be a random variable with expectation $\mu_X = 2$. What is the expectation of $5X + 2$?

   Solution:
   $$5\mu_X + 2 = 12.$$

2. **Sums of Independent Random Variables.** Let X and Y be random variables with $\mu_X = 1$, $\mu_Y = -5$. What is $\mathrm{E}(X + Y)$?

   Solution:
   $$\mathrm{E}(X + Y) = \mu_X + \mu_Y = 1 + (-5) = -4.$$

3. Let X and Y be random variables with $\mu_X = -2$, $\mu_Y = 3$.
   (a) Find the expectation of $-3X + 2$.

   Solution:
   $$\mathrm{E}(-3X + 2) = -3\mu_X + 2 = -3(-2) + 2 = 8.$$

   (b) Find the expectation of $X + Y$.

   Solution:
   $$\mathrm{E}(X + Y) = \mu_X + \mu_Y = 1.$$

4. You invite four people to go out to dinner on Friday night. The attendance probabilities for the four potential guests are 50%, 20%, 30%, and 90%.
   (a) Find the expected number of guests.

   Solution: Let X be the number of guests. Then, X can be written as
   $$X = Y_1 + Y_2 + Y_3 + Y_4,$$
   where
   $$Y_i = \begin{cases} 1 & \text{if guest } i \text{ attends,} \\ 0 & \text{otherwise.} \end{cases}$$

Then, $\mathrm{E}[Y_1] = .50$, $\mathrm{E}[Y_2] = .20$, $\mathrm{E}[Y_3 = .30]$, and $\mathrm{E}[Y_4] = .90$, so

$$\begin{aligned}\mathrm{E}[X] &= \mathrm{E}[Y_1 + Y_2 + Y_3 + Y_4] \\ &= \mathrm{E}[Y_1] + \mathrm{E}[Y_2] + \mathrm{E}[Y_3] + \mathrm{E}[Y_4] \\ &= .50 + .20 + .30 + .90 \\ &= 1.9.\end{aligned}$$

(b) The dinner will be a *prix fixe* meal, costing \$50 per person. What is the expected total cost for yourself and your guests?

Solution: The total cost is $C = 50 + 50X$. Thus,

$$\begin{aligned}\mathrm{E}[C] &= \mathrm{E}[50 + 50X] \\ &= 50 + 50\,\mathrm{E}[X] \\ &= 50 + 50(1.9) \\ &= 145.\end{aligned}$$

The expected total cost is \$145.

(c) What is the interpretation of your answer to part (b)?

Solution: If there were many similar nights with the same circumstances, then the average cost of all of the dinners would be \$145.

Binomial Random Variables

5. A certain coin has a 25% of landing heads, and a 75% chance of landing tails.

   (a) If you flip the coin 4 times, what is the chance of getting exactly 2 heads?

   Solution: There are 6 outcomes whith exactly 2 heads:

   $$HHTT, HTHT, HTTH, THHT, THTH, TTHH.$$

   By independence, each of these outcomes has probability $(.25)^2(.75)^2$. Thus,

   $$\text{P(exactly 2 heads out of 4 flips)} = 6(.25)^2(.75)^2.$$

   (b) If you flip the coin 10 times, what is the chance of getting exactly 2 heads?

   Solution: Rather than list all outcomes, we will use a counting rule. There are ${}_{10}C_2$ ways of choosing the positions for the two heads; each of these outcomes has probability $(.25)^2(.75)^8$. Thus,

   $$\text{P(exactly 2 heads out of 10 flips)} = {}_{10}C_2\,(.25)^2(.75)^8.$$

6. Suppose that you are rolling a die eight times. Find the probability that the face with two spots comes up exactly twice.

Solution: Let X be the number of times that we get the face with two spots. This is a binomial random variable with $n = 8$ and $p = \frac{1}{6}$. We compute

$$\begin{aligned} P(X = 2) &= {}_nC_2\,p^2(1-p)^{n-2} \\ &= {}_8C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^6 \\ &\approx 0.26. \end{aligned}$$

7. The probability is 0.04 that a person reached on a "cold call" by a telemarketer will make a purchase. If the telemarketer calls 40 people, what is the probability that at least one sale with result?

Solution: Let X be the number of sales. This is a binomial random variable with $n = 40$ and $p = 0.04$. Thus,

$$\begin{aligned} P(X \geq 1) &= 1 - P(X < 1) \\ &= 1 - P(X = 0) \\ &= 1 - {}_nC_0\, p^0(1-p)^{n-0} \\ &= 1 - (0.96)^{40} \\ &\approx .805 \end{aligned}$$

8. A new restaurant opening in Greenwich village has a 30% chance of survival during their first year. If 16 restaurants open this year, find the probability that

(a) exactly 3 restaurants survive.

Solution: Let X be the number that survive. This is a binomial random variable with $n = 16$ and $p = 0.3$. Therefore,

$$\begin{aligned} P(X = 3) &= {}_{16}C_3\,(0.3)^3(1-0.3)^{(16-3)} \\ &= .146 \end{aligned}$$

(b) fewer than 3 restaurants survive.

Solution:

$$\begin{aligned} P(X < 3) &= P(X = 0) + P(X = 1) + P(X = 2) \\ &= {}_{16}C_0\,(0.3)^0(0.7)^{16} + {}_{16}C_1\,(0.3)^1(0.7)^{15} + {}_{16}C_2\,(0.3)^2(0.7)^{14} \\ &= .099 \end{aligned}$$

(c) more than 3 restaurants survive.

Solution:

$$\begin{aligned} P(X > 3) &= 1 - P(X \leq 3) \\ &= 1 - (.099 + .146) \\ &= .754 \end{aligned}$$

9. The probability of winning at a certain game is 0.10. If you play the game 10 times, what is the probability that you win at most once?

Solution: Let X be the number of times that we win. This is a binomial random variable with $n = 10$ and $p = 0.10$. We compute

$$\begin{aligned} P(X \le 1) &= P(X = 0) + P(X = 1) \\ &= {}_nC_0\, p^0(1-p)^{n-0} + {}_nC_1\, p^1(1-p)^{n-1} \\ &= {}_{10}C_0\,(0.10)^0(0.90)^{10} + {}_{10}C_1\,(0.10)^1(0.90)^9 \\ &= (0.90)^{10} + 10\,(0.10)(0.90)^9 \\ &\approx 0.736. \end{aligned}$$

10. The probability is 0.2 that an audit of a retail business will turn up irregularities in the collection of state sales tax. If 20 retail businesses are audited, find the probability that

(a) fewer than 2 will have irregularities in the collection of state sales tax.

Solution: Let X be the number audited. This is a binomial random variable with $n = 20$ and $p = 0.2$. Therefore,

$$P(X < 2) = {}_{20}C_0\,(0.2)^0(0.8)^{20} + {}_{20}C_1\,(0.2)^1(0.8)^{19} \qquad \approx .069$$

(b) more than 2 will have irregularities in the collection of state sales tax.

Solution:

$$\begin{aligned} P(X > 2) &= 1 - P(X \le 2) \\ &= 1 - \left[{}_{20}C_0\,(0.2)^0(0.8)^{20} + {}_{20}C_1\,(0.2)^1(0.8)^{19} + {}_{20}C_2\,(0.2)^2(0.8)^{18}\right] \\ &\approx .794. \end{aligned}$$