The Central Limit Theorem – Solutions

COR1-GB.1305 – Statistics and Data Analysis

1. Consider the population of all Fortune 500 CEOs and their salaries. Suppose that the mean salary (in millions of dollars) is $\mu = 20$, and the standard deviation of the salaries is $\sigma = 5$. You sample 50 CEOs and find their salaries.

   (a) Draw a histogram of what you think the population looks like.

   (b) Consider the sample mean $\bar{X}$ to be a random variable. What is the expectation of $\bar{X}$?

   Solution: $\mathrm{E}[\bar{X}] = \mu = 20$.

   (c) What is the standard deviation of $\bar{X}$?

   Solution: $\mathrm{sd}[\bar{X}] = \frac{\sigma}{\sqrt{n}} = \frac{5}{\sqrt{50}} \approx .707$.

   (d) Draw a picture of what you think the PDF of $\bar{X}$ looks like.

   Solution: Normal with mean 20, standard deviation .707.

2. You draw a random sample of size $n = 64$ from a population with mean $\mu = 50$ and standard deviation $\sigma = 16$. From this, you compute the sample mean, $\bar{X}$.

   (a) What are the expectation and standard deviation of $\bar{X}$?

   Solution:

   $$\begin{aligned} \mathrm{E}[\bar{X}] &= \mu = 50, \\ \mathrm{sd}[\bar{X}] &= \frac{\sigma}{\sqrt{n}} = \frac{16}{\sqrt{64}} = 2. \end{aligned}$$

   (b) Approximately what is the probability that the sample mean is above 54?

   Solution: The sample mean has expectation 50 and standard deviation 2. By the central limit theorem, the sample mean is approximately normally distributed. Thus, by the empirical rule, there is roughly a 2.5% chance of being above 54 (2 standard deviations above the mean).

   (c) Do you need any additional assumptions for part (c) to be true?

   Solution: No. Since the sample size is large ($n \geq 30$), the central limit theorem applies.

3. You draw a random sample of size $n = 16$ from a population with mean $\mu = 100$ and standard deviation $\sigma = 20$. From this, you compute the sample mean, $\bar{X}$.

   (a) What are the expectation and standard deviation of $\bar{X}$?

   Solution:

   $$\begin{aligned} \mathrm{E}[\bar{X}] &= \mu = 100, \\ \mathrm{sd}[\bar{X}] &= \frac{\sigma}{\sqrt{n}} = \frac{20}{\sqrt{16}} = 5. \end{aligned}$$

   (b) Approximately what is the probability that the sample mean is between 95 and 105?

   Solution: The sample mean has expectation 100 and standard deviation 5. If it is approximately normal, then we can use the empirical rule to say that there is a 68% of being between 95 and 105 (within one standard deviation of its expecation).

   (c) Do you need any additional assumptions for part (c) to be true?

   Solution: Yes, we need to assume that the population is normal. The sample size is small ($n < 30$), so the central limit theorem may not be in force.