Binomial Random Variables

1. A certain coin has a 25% of landing heads, and a 75% chance of landing tails.
   (a) If you flip the coin 4 times, what is the chance of getting exactly 2 heads?

   **Solution:** There are 6 outcomes with exactly 2 heads:
   
   \[ HHTT, HTHT, HTTH, THHT, THTH, TTHH. \]
   
   By independence, each of these outcomes has probability \((.25)^2(.75)^2\). Thus,
   
   \[ P(\text{exactly 2 heads out of 4 flips}) = 6(.25)^2(.75)^2. \]

   (b) If you flip the coin 10 times, what is the chance of getting exactly 2 heads?

   **Solution:** Rather than list all outcomes, we will use a counting rule. There are \(10C_2\) ways of choosing the positions for the two heads; each of these outcomes has probability \((.25)^2(.75)^8\). Thus,
   
   \[ P(\text{exactly 2 heads out of 10 flips}) = 10C_2 (.25)^2(.75)^8. \]

2. Suppose that you are rolling a die eight times. Find the probability that the face with two spots comes up exactly twice.

   **Solution:** Let \(X\) be the number of times that we get the face with two spots. This is a binomial random variable with \(n = 8\) and \(p = \frac{1}{6}\). We compute
   
   \[
   P(X = 2) = nC_2 p^2(1 - p)^{n-2}
   
   = 8C_2 \left( \frac{1}{6} \right)^2 \left( \frac{5}{6} \right)^6
   
   \approx 0.26.
   \]

3. The probability is 0.04 that a person reached on a “cold call” by a telemarketer will make a purchase. If the telemarketer calls 40 people, what is the probability that at least one sale with result?
**Solution:** Let $X$ be the number of sales. This is a binomial random variable with $n = 40$ and $p = 0.04$. Thus,

\[
P(X \geq 1) = 1 - P(X < 1) = 1 - P(X = 0) = 1 - \binom{n}{0} p^0 (1 - p)^{n-0} = 1 - (0.96)^{40} \approx 0.805
\]
4. A new restaurant opening in Greenwich village has a 30% chance of survival during their first year. If 16 restaurants open this year, find the probability that exactly 3 restaurants survive.

**Solution:** Let $X$ be the number that survive. This is a binomial random variable with $n = 16$ and $p = 0.3$. Therefore,

$$P(X = 3) = 16C_3 (0.3)^3(1 - 0.3)^{16 - 3}$$

$$= .146$$

5. The probability of winning at a certain game is 0.10. If you play the game 10 times, what is the probability that you win at most once?

**Solution:** Let $X$ be the number of times that we win. This is a binomial random variable with $n = 10$ and $p = 0.10$. We compute

$$P(X \leq 1) = P(X = 0) + P(X = 1)$$

$$= \binom{n}{0} p^0(1 - p)^{n-0} + \binom{n}{1} p^1(1 - p)^{n-1}$$

$$= 10C0 (0.10)^0(0.90)^{10} + 10C1 (0.10)^1(0.90)^9$$

$$= (0.90)^{10} + 10 (0.10)(0.90)^9$$

$$\approx 0.736$$

6. The probability is 0.3 that an audit of a retail business will turn up irregularities in the collection of state sales tax. If 16 retail businesses are audited, find the probability that

(a) fewer than 3 will have irregularities in the collection of state sales tax.

**Solution:** Let $X$ be the number audited. This is a binomial random variable with $n = 16$ and $p = 0.3$. Therefore,

$$P(X < 3) = 16C0 (0.3)^0(0.7)^{16} + 16C1 (0.3)^1(0.7)^{15} + 16C2 (0.3)^2(0.7)^{14}$$

$$\approx .0994.$$ 

(b) more than 3 will have irregularities in the collection of state sales tax.

**Solution:**

$$P(X > 3) = 1 - P(X \leq 3)$$

$$= 1 - \left[16C0 (0.3)^0(0.7)^{16} + 16C1 (0.3)^1(0.7)^{15} + 16C2 (0.3)^2(0.7)^{14} + 16C3 (0.3)^3(0.7)^{13}\right]$$

$$\approx .7541.$$
Poisson Random Variables

7. The number of calls arriving at the Swampside Police Station follows a Poisson distribution with rate 4.6/hour.
   
   (a) What is the probability that exactly six calls will come between 8:00 p.m. and 9:00 p.m.?

   **Solution:** Let $X$ be the number of calls that arrive between 8:00 p.m. and 9:00 p.m. This is a Poisson random variable with mean
   
   $$\lambda = E(X) = (4.6 \text{ calls/hour})(1 \text{ hour}) = 4.6 \text{ calls}.$$  
   
   Thus,
   
   $$P(X = 6) = \frac{\lambda^6}{6!} e^{-\lambda} = \frac{(4.6)^6}{6!} e^{-4.6}.$$  

   (b) Find the probability that exactly 7 calls will come between 9:00 p.m. and 10:30 p.m.

   **Solution:** Let $X$ be the number of calls that arrive between 9:00 p.m. and 10:30 p.m. This is a Poisson random variable with mean
   
   $$\lambda = E(X) = (4.6 \text{ calls/hour})(1.5 \text{ hours}) = 6.9 \text{ calls}.$$  
   
   Thus,
   
   $$P(X = 7) = \frac{\lambda^7}{7!} e^{-\lambda} = \frac{(6.9)^7}{7!} e^{-6.9}.$$  

8. Car accidents occur at a particular intersection in the city at a rate of about 2/year.
   
   (a) Estimate the probability of no accidents occurring in a 6-month period.

   **Solution:** Let $X$ be the number of car accidents. This is Poisson random variable with mean
   
   $$\lambda = E(X) = (2 \text{ accidents/year})(0.5 \text{ years}) = 1 \text{ accident}.$$  
   
   Thus,
   
   $$P(X = 0) = \frac{\lambda^0}{0!} e^{-\lambda} = e^{-1} \approx 0.368.$$  

   (b) Estimate the probability of two or more accidents occurring in a year.

   **Solution:** Let $X$ be the number of car accidents. This is Poisson random variable with mean
   
   $$\lambda = E(X) = (2 \text{ accidents/year})(1.0 \text{ years}) = 2 \text{ accident}.$$
Thus,

\[ P(X \geq 2) = 1 - P(X < 2) \]
\[ = 1 - \left[ \frac{\lambda^0}{0!} e^{-\lambda} + \frac{\lambda^1}{1!} e^{-\lambda} \right] \]
\[ \approx .594. \]
Empirical Rule with Binomial and Poisson Random Variables

9. If you flip a fair coin 100 times, would it be unusual to get 42 heads and 58 tails?

**Solution:** Let $X$ be the number of heads. Then, $X$ is binomial with $n = 100$ and $p = 0.5$. Thus, its expectation and standard deviation are

$$
\mu = np = (100)(0.5) = 50,
$$

and

$$
\sigma = \sqrt{np(1-p)} = \sqrt{(100)(0.5)(1-0.5)} = 5.
$$

Since $np \geq 15$ and $n(1-p) \geq 15$, we can use the empirical rule to approximate the distribution of $X$. Thus, approximately 95% of the time, $X$ will be in the range $\mu \pm 2\sigma$, or $(40, 60)$. So, it would not be unusual to observe $X = 42$.

10. If $X$ is a Poisson random variable with $\lambda = 225$, would it be unusual to get a value of $X$ which is less than 190?

**Solution:** Set

$$
\mu = E(X) = \lambda = 255,
$$

$$
\sigma = sd(X) = \sqrt{\lambda} = 15.
$$

Define $z$ to be the number of standard deviations above the mean that 190 is, i.e.

$$
190 = \mu + \sigma z.
$$

Then,

$$
z = \frac{190 - \mu}{\sigma} = \frac{-35}{15} \approx -2.33.
$$

A value of $X$ which is below 190 is more than 2.33 standard deviations below the mean of $X$. The empirical rule tells us that observations more than 2 standard deviations away from the mean are unusual (they occur less than 95% of the time). Therefore, values of $X$ below 190 are unusual.

11. The probability is 0.10 that a person reached on a “cold call” by a telemarketer will make a purchase. If the telemarketer calls 200 people, would it be unusual for them to get 30 purchases?
Solution: Let $X$ be the number of purchases. This is a Binomial random variable with size $n = 200$ and success probability $p = 0.10$. Thus, the expectation and standard deviation of $X$ are

$$
\mu = np = (200)(.10) = 20
$$

$$
\sigma = \sqrt{np(1-p)} = \sqrt{(200)(.10)(.90)} = \sqrt{18} \approx 4.2
$$

Since $np \geq 15$ and $np(1 - p) \geq 15$, the distribution of $X$ can be approximated by the empirical rule. Using the empirical rule approximation, 95% of the time, $X$ will be in the range $\mu \pm 2\sigma$, or $(11.6, 28.4)$, and 99.7% of the time, $X$ will be in the range $\mu \pm 3\sigma$, or $(7.4, 32.6)$. We would see $X \geq 30$ less than 5% of the time. It would be unusual to see $X = 30$, but not highly unusual.