Binomial random variables

1. A certain coin has a 25% of landing heads, and a 75% chance of landing tails.
   (a) If you flip the coin 4 times, what is the chance of getting exactly 2 heads?

   **Solution:** There are 6 outcomes with exactly 2 heads:
   \[ HHTT, HTHT, HTTH, THHT, THTH, TTHH. \]
   By independence, each of these outcomes has probability \((.25)^2(.75)^2\). Thus,
   \[
P(\text{exactly 2 heads out of 4 flips}) = 6(.25)^2(.75)^2.
   \]

   (b) If you flip the coin 10 times, what is the chance of getting exactly 2 heads?

   **Solution:** Rather than list all outcomes, we will use a counting rule. There are \(\binom{10}{2}\) ways of choosing the positions for the two heads; each of these outcomes has probability \((.25)^2(.75)^8\). Thus,
   \[
P(\text{exactly 2 heads out of 10 flips}) = \binom{10}{2}(.25)^2(.75)^8.
   \]

2. Suppose that you are rolling a die eight times. Find the probability that the face with two spots comes up exactly twice.

   **Solution:** Let \(X\) be the number of times that we get the face with two spots. This is a binomial random variable with \(n = 8\) and \(p = \frac{1}{6}\). We compute
   \[
P(X = 2) = \binom{n}{2}p^2(1-p)^{n-2} = \binom{8}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^6 
   \approx 0.26.
   \]
3. The probability is 0.04 that a person reached on a “cold call” by a telemarketer will make a purchase. If the telemarketer calls 40 people, what is the probability that at least one sale will result?

**Solution:** Let $X$ be the number of sales. This is a binomial random variable with $n = 40$ and $p = 0.04$. Thus,

\[
P(X \geq 1) = 1 - P(X < 1) = 1 - P(X = 0) = 1 - \binom{n}{0} p^0 (1 - p)^{n-0} = 1 - (0.96)^{40} \approx 0.805
\]
4. A new restaurant opening in Greenwich village has a 30% chance of survival during their first
year. If 16 restaurants open this year, find the probability that

(a) exactly 3 restaurants survive.

**Solution:** Let $X$ be the number that survive. This is a binomial random variable
with $n = 16$ and $p = 0.3$. Therefore,

$$P(X = 3) = \binom{16}{3}(0.3)^3(1 - 0.3)^{16 - 3}$$
$$= .146$$

(b) fewer than 3 restaurants survive.

**Solution:**

$$P(X < 3) = \binom{16}{0}(0.3)^0(0.7)^{16} + \binom{16}{1}(0.3)^1(0.7)^{15} + \binom{16}{2}(0.3)^2(0.7)^{14}$$
$$= .099.$$ 

(c) more than 3 restaurants survive.

**Solution:**

$$P(X > 3) = 1 - P(X \leq 3)$$
$$= 1 - (.099 + .146)$$
$$= .754.$$
Poisson random variables

5. The number of calls arriving at the Swampside Police Station follows a Poisson distribution with rate 4.6/hour. What is the probability that exactly six calls will come between 8:00 p.m. and 9:00 p.m.?

**Solution:** Let $X$ be the number of calls that arrive between 8:00 p.m. and 9:00 p.m. This is a Poisson random variable with mean

$$\lambda = E(X) = (4.6 \text{ calls/hour})(1 \text{ hour}) = 4.6 \text{ calls}.$$ 

Thus,

$$P(X = 6) = \frac{\lambda^6}{6!} e^{-\lambda} = \frac{(4.6)^6}{6!} e^{-4.6}.$$ 

6. In the station from Problem 5, find the probability that exactly 7 calls will come between 9:00 p.m. and 10:30 p.m.

**Solution:** Let $X$ be the number of calls that arrive between 9:00 p.m. and 10:30 p.m. This is a Poisson random variable with mean

$$\lambda = E(X) = (4.6 \text{ calls/hour})(1.5 \text{ hours}) = 6.9 \text{ calls}.$$ 

Thus,

$$P(X = 7) = \frac{\lambda^7}{7!} e^{-\lambda} = \frac{(6.9)^7}{7!} e^{-6.9}.$$ 

7. Car accidents occur at a particular intersection in the city at a rate of about 2/year. Estimate the probability of no accidents occurring in a 6-month period.

**Solution:** Let $X$ be the number of car accidents. This is Poisson random variable with mean

$$\lambda = E(X) = (2 \text{ accidents/year})(0.5 \text{ years}) = 1 \text{ accident}.$$ 

Thus,

$$P(X = 0) = \frac{\lambda^0}{0!} e^{-\lambda} = e^{-1} \approx .368.$$
8. In the intersection from Problem 7, estimate the probability of two or more accidents occurring in a year.

**Solution:** Let $X$ be the number of car accidents. This is Poisson random variable with mean

$$\lambda = E(X) = (2 \text{ accidents/year})(1.0 \text{ years}) = 2 \text{ accident}.$$ 

Thus,

$$P(X \geq 2) = 1 - P(X < 2)$$

$$= 1 - \left[ \frac{\lambda^0}{0!} e^{-\lambda} + \frac{\lambda^1}{1!} e^{-\lambda} \right]$$

$$\approx .594.$$
Empirical rule with Binomial and Poisson random variables

9. If \( X \) is a Poisson random variable with \( \lambda = 225 \), would it be unusual to get a value of \( X \) which is less than 190?

**Solution:** Set

\[
\mu = E(X) = \lambda = 255, \\
\sigma = \text{sd}(X) = \sqrt{\lambda} = 15.
\]

Define \( z \) to be the number of standard deviations above the mean that 190 is, i.e.

\[
190 = \mu + \sigma z.
\]

Then,

\[
z = \frac{190 - \mu}{\sigma} = \frac{-35}{15} \approx -2.33.
\]

A value of \( X \) which is below 190 is more than 2.33 standard deviations below the mean of \( X \). The empirical rule tells us that observations more than 2 standard deviations away from the mean are unusual (they occur less than 95% of the time). Therefore, values of \( X \) below 190 are unusual.

10. The probability is 0.10 that a person reached on a “cold call” by a telemarketer will make a purchase. If the telemarketer calls 200 people, would it be unusual for them to get 30 purchases?

**Solution:** Let \( X \) be the number of purchases. This is a Binomial random variable with size \( n = 200 \) and success probability \( p = 0.10 \). Thus, the expectation and standard deviation of \( X \) are

\[
\mu = np = (200)(.10) = 20 \\
\sigma = \sqrt{np(1-p)} = \sqrt{(200)(.10)(.90)} = \sqrt{18} \approx 4.2
\]

Since \( np \geq 15 \) and \( np(1-p) \geq 15 \), the distribution of \( X \) can be approximated by the empirical rule. Using the empirical rule approximation, 95% of the time, \( X \) will be in the range \( \mu \pm 2\sigma \), or \((11.6, 28.4)\), and 99.7% of the time, \( X \) will be in the range \( \mu \pm 3\sigma \), or \((7.4, 32.6)\). We would see \( X \geq 30 \) less than 5% of the time. It would be unusual to see \( X = 30 \), but not highly unusual.