p-values / Tests on Proportions – Solutions

STAT-UB.0103 – Statistics for Business Control and Regression Models

p-values

1. In the "Quarter Pounder" example, we we tested the null hypothesis that the weight of a McDonald's quarter pounder is 0.25 pounds ($H_0 : \mu = 0.25$) against the alternative that the weight is below 0.25 pounds ($H_a : \mu < 0.25$). After collecting a sample our observed z statistic was $z = -2.02$. Find the largest level α at which the hypothesis testing procedure does not reject H_0.

Solution: This problem has hypotheses

$$H_0 : \mu = 0.25,$$
$$H_a : \mu < 0.25.$$

We reject H_0 for small values of the Z statistic, i.e. we reject when $Z < -z_\alpha$. The smallest level α at which we could perform a test and still reject H_0 is the value at which $z = -z_\alpha$. Since $\alpha = \Phi(-z_\alpha)$, we have

$$\alpha = \Phi(z) = 2.17\%.$$

2. Suppose we perform a hypothesis test and we observe a p-value of $p = .02$. True or false: There is a 2% chance that the null hypothesis is true.

Solution: False. The p-value is the probability of getting a test statistic at least as extreme as what was observed. Heuristically, we can think of this as

$$\mathrm{P}(\text{Data} \mid H_0 \text{ is true}) = 2\%.$$

The statement in the problem is

$$\mathrm{P}(H_0 \text{ is true} \mid \text{Data}) = 2\%.$$

Clearly, this is not the same.

3. Suppose we perform a hypothesis test and we observe a p-value of $p = .02$. True or false: If we reject the null hypothesis, then there is a 2% chance of making a type I error.

Solution: False. We can only make a type I error when the null hypothesis is true. Thus, the statement in question 3 is *exactly the same* as the statement in question 2.

4. Suppose we perform a hypothesis test and we observe a Z test statistic $z = -2.02$, corresponding to a p-value of $p = .02$. True or false: If we were to repeat the experiment and the null hypothesis were actually true, then there would be a 2% chance of observing a test statistic at least as extreme as $z = -2.02$.

> **Solution:** True. The p-value is the probability of getting a test statistic as least as extreme as the observed value if the null hypothesis were true. Note: for a one-sided less-than alternative, extreme means "less than or equal to."

Tests on a population proportion

5. Suppose you have a population with an unknown proportion p of successes. You want to test the null hypothesis $H_0 : p = 0.2$ against the alternative $H_a : p \neq 0.2$. To this end, you collect a sample of size $n = 100$. It terns out that there are $x = 30$ successes in the sample, so that the sample proportion is $\hat{p} = \frac{30}{100} = .30$. Is there sufficient evidence to reject the null hypothesis at level 5%?

> **Solution:** We use a large sample test on the population proportion. The test statistic is
>
> $$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{.3 - .2}{\sqrt{(.2)(.8)/100}} = 2.5$$
>
> The rejection region is $|z| > z_{\alpha/2}$ with $\alpha = .05$. So, the rejection region is $|z| > 2$ (or, more precisely, $|z| > 1.96$). Since $|2.5| > 2$, we reject H_0. There is significant evidence that the true population proportion is not equal to .20.

6. Suppose you have a population with an unknown proportion p of successes. You want to test the null hypothesis $H_0 : p = 0.6$ against the alternative $H_a : p > 0.6$. To this end, you collect a sample of size $n = 81$ It terns out that there are $x = 53$ successes in the sample, so that the sample proportion is $\hat{p} = \frac{53}{81} = .654$. You want to perform a hypothesis test at level 5%.

   (a) What is the test statistic?

   > **Solution:** We use a large sample test on the population proportion. The test statistic is
   >
   > $$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{.65 - .6}{\sqrt{(.6)(.4)/81}} \approx 1.1$$

   (b) What is the rejection region?

   > **Solution:** Since we have a one-sided greater-than alternative, The rejection region is $z > z_\alpha$ with $\alpha = .05$. So, the rejection region is $z > 1.645$

   (c) What is the result of the hypothesis test?

   > **Solution:** Since $|1.1| \leq 1.645$, we do not reject H_0. There is not significant evidence that the true population proportion is not equal to .60.

7. In a May 2006 random-digit-dialiing telephone survey of 4,000 American adults, 42% of the sample had access to a high-speed internet connection at home. Let p represent the true proportion of all American adults who had access to a high-speed internet connection at home in 2006.

   In 2005, the Pew Internet & American Life Project reported that 30% of all American adults had access to a high-speed internet connection.

   Perform a test at significance level 5% of whether the proportion changed in 2006.

   (a) What are the population and the sample?

   > **Solution:** Population: high-speed internet statuses of all Americans in 2006.
   > Sample: high-speed internet statuses of the 4,000 telephoned households.

   (b) What are the null and alternative hypotheses?

   > **Solution:**
   >
   > $$H_0 : p = .30 \qquad \text{(no change)}$$
   >
   > $$H_a : p \neq .30 \qquad \text{(change)}$$

   (c) What is the test statistic?

   > **Solution:** We use a z test for the proportion:
   >
   > $$Z = \frac{\hat{P} - .30}{\sqrt{.30(1 - .30)/n}}.$$

   (d) What is the rejection region?

   > **Solution:** We reject the null hypothesis if $|Z| > 2$. (Or, more precisely, reject if $|Z| > 1.96$.)

   (e) What assumptions are you making?

   > **Solution:** That the sample is unbiased.

   (f) What is the result of the test?

Solution: The observed z-statistic is

$$\begin{aligned} z &= \frac{.42 - .30}{\sqrt{.30(1-.30)/4000}} \\ &\approx 16.5. \end{aligned}$$

Since $|z| > 2$, we reject H_0.

8. An alkaline battery manufacturer wants to be reasonably certain that fewer than 5% of its batteries are defective. Suppose 300 batteries are randomly selected from a very large shipment; each is tested, and 10 defective batteries are found. Does this provide sufficient evidence for the manufacturer to conclude that the fraction defective in the entire shipment is less than .05? Use a significance level of $\alpha = .01$.

   (a) What are the population and the sample?

   Solution: Population: all batteries in the shipment and whether or not they are defective.
   Sample: the 300 sampled batteries.

   (b) To err on the side of caution, the manufacturer takes the null hypothesis to be that the shipment has a high proportion of defective batteries. In terms of the population parameter, what are the null and alternative hypotheses?

   Solution:

$$H_0 : p = .05 \qquad \text{(high defect rate)}$$

$$H_a : p < .05 \qquad \text{(low defect rate)}$$

   (c) What is the test statistic?

   Solution: We use a z test for the proportion:

$$Z = \frac{\hat{P} - .05}{\sqrt{.05(1 - .05)/n}}.$$

   (d) What is the rejection region?

   Solution: We reject the null hypothesis if $Z < -z_\alpha$. (Or, more precisely, reject if $z < -z_{.01} = -2.326$.)

   (e) What assumptions are you making?

   Solution: That the sample is unbiased.

(f) What is the result of the test? Is there sufficient evidence to conclude that the defect rate is acceptable?

Solution: We have that

$$\hat{p} = \frac{10}{300} \approx .0333$$

The observed z-statistic is

$$\begin{aligned} z &= \frac{.0333 - .05}{\sqrt{.05(1 - .05)/300}} \\ &\approx -1.32. \end{aligned}$$

Since $z \geq -2.325$, we do not reject H_0. There is not sufficient evidence to conclude that the defect rate is below 5%.