Introduction to Probability – Solutions

STAT-UB.0103 – Statistics for Business Control and Regression Models

Sample Points and Sample Spaces

1. In the following two experiments, what are the sample points and the sample space?

   (a) You flip a coin.

   > **Solution:** The sample points are H, "the outcome is heads," and T, "the outcome is tails." The sample space is the set of all sample points: $\Omega = \{H, T\}$.

   (b) You roll a 6-sided die.

   > **Solution:** The sample points are the possible outcomes of the die: 1, 2, 3, 4, 5, 6. The sample space is the set of all sample points: $\Omega = \{1, 2, 3, 4, 5, 6\}$.

2. Suppose that a customer visits a restaurant and leaves a review on Yelp with 1–5 stars. What are the sample points and the sample space for the customer's star rating?

> **Solution:** The sample points are the possible star ratings: 1, 2, 3, 4, 5. The sample space is the set of all sample points: $\Omega = \{1, 2, 3, 4, 5\}$.

3. Suppose that two customers visit a restaurant, and that they both leave Yelp reviews with 1–5 stars each. What are the sample points and the sample space for the pair of star ratings?

> **Solution:** Each sample point can be represented by an ordered pair (i, j), where i is the first customer's star rating, and j is second customer's star rating. The sample points are the elements of the following table.

	1	2	$\cdots$	5
1	(1,1)	(1,2)	$\cdots$	(1,5)
2	(2,1)	(2,2)	$\cdots$	(2,5)
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
5	(5,1)	(5,2)	$\cdots$	(5,5)

> The sample space is the set of all 25 sample points: $\Omega = \{(1, 1), (1, 2), \ldots, (5, 5)\}$.

4. Suppose you randomly pick a respondent from the class survey, then record their major and gender. What are the sample points and the sample space?

Solution: Each sample point is a major-gender pair. The sample space is the set of all possible major-gender pairs: $\Omega = \{(\text{Accounting}, \text{Female}),\ (\text{Accounting}, \text{Male}),\ (\text{Art}, \text{Female}),\ (\text{Art}, \text{Male}), \dots\}$. Note that the sample space is all *possible* major-gender pairs, not all *observed* major-gender pairs.

Events

5. Suppose that a customer leaves a Yelp rating (1–5 stars) for a restaurant. Describe the event "the rating is odd (not even)."

> **Solution:**
> $$O = \{1, 3, 5\}.$$

6. Suppose you randomly pick a respondent from the class survey, then record their major and gender. Assume that gender is listed as Male or Female, and that major is listed as Finance, Other, or Undecided.

   (a) Describe the event "the major is Undecided."

   > **Solution:** $U = \{(\text{Undecided}, \text{Female}), (\text{Undecided}, \text{Male})\}$.

   (b) Describe the event "the gender is Male."

   > **Solution:** $M = \big\{(\text{Finance}, \text{Male}), (\text{Other}, \text{Male}), (\text{Undecided}, \text{Male})\big\}$.

Probability

7. Suppose you randomly pick a respondent from the class survey and record their major and gender.

   (a) Use the following table of recorded survey response frequencies to compute the probabilities of the sample points.

	Gender		
Major	Female	Male	Total
Finance	12	20	32
Other	4	3	7
Undecided	10	15	25
Total	26	38	66

Solution: To compute the probabilities for the 6 sample points corresponding to the cells of the table, we take the recorded frequency and divide by the total number of survey respondents. We have

$$\begin{aligned}
\mathrm{P}\left(\text{Finance}, \text{Female}\right) &= \frac{12}{66}, \\
\mathrm{P}\left(\text{Finance}, \text{Male}\right) &= \frac{20}{66}, \\
\mathrm{P}\left(\text{Other}, \text{Female}\right) &= \frac{4}{66}, \\
\mathrm{P}\left(\text{Other}, \text{Male}\right) &= \frac{3}{66}, \\
\mathrm{P}\left(\text{Undecided}, \text{Female}\right) &= \frac{10}{66}, \\
\mathrm{P}\left(\text{Undecided}, \text{Female}\right) &= \frac{15}{66}.
\end{aligned}$$

(b) Find the probabilities of the events in problem 6.

Solution:

$$\begin{aligned}
\mathrm{P}(U) &= \tfrac{10}{66} + \tfrac{15}{66} \\
&= \tfrac{25}{66} \\
&= 38\%,
\end{aligned}$$

$$\begin{aligned}
\mathrm{P}(M) &= \tfrac{3}{66} + \tfrac{15}{66} + \tfrac{20}{66} \\
&= \tfrac{38}{66} \\
&= 57\%.
\end{aligned}$$

8. Suppose that a customers Yelp rating is random, and that the probabilities for the possible star ratings are $p_1 = 10\%$, $p_2 = 30\%$, $p_3 = 25\%$, $p_4 = 20\%$, $p_5 = 15\%$. Find the probability of the event in problem 5.

> **Solution:** We add up the probabilities of the sample points in the event:
>
> $$\begin{aligned} \mathrm{P}(\{1,3,5\}) &= p_1 + p_3 + p_5 \\ &= 10\% + 25\% + 15\% \\ &= 50\%. \end{aligned}$$

Compound Events and the Additive Rule

9. Suppose you pick a random survey respondent and record their major and gender.

   (a) Describe the event "the major is Undecided or the gender is Male."

   > **Solution:**
   > Denote the event by E. There are many ways to describe E. One way is just to list the sample points in E:
   >
   > $$E = \{(\text{Undecided}, \text{Female}), (\text{Undecided}, \text{Male}), (\text{Finance}, \text{Male}), (\text{Other}, \text{Male})\}.$$
   >
   > Another way is to describe E by a union:
   >
   > $$\begin{aligned} E &= \{\text{major is Undecided}\} \cup \{\text{gender is Male}\} \\ &= U \cup M. \end{aligned}$$

   (b) Compute the probability of the event in part (a) by adding the probabilities of all of the sample points in the event.

   > **Solution:**
   >
   > $$\begin{aligned} \mathrm{P}(E) &= \tfrac{10}{66} + \tfrac{15}{66} + \tfrac{20}{66} + \tfrac{3}{66} \\ &= \tfrac{48}{66} \\ &= 73\%. \end{aligned}$$

   (c) Compute the probability of the event in part (a) by using the additive rule.

Solution:

$$\begin{aligned}
\mathrm{P}(E) &= \mathrm{P}(U \cup M) \\
&= \mathrm{P}(U) + \mathrm{P}(M) - \mathrm{P}(U \cap M) \\
&= \tfrac{25}{66} + \tfrac{38}{66} - \tfrac{15}{66} \\
&= \tfrac{48}{66} \\
&= 73\%.
\end{aligned}$$

10. Suppose that two customers give ratings (1–5 stars) to the same restaurant on Yelp.

(a) Describe the event "at least one customer gives a 1 star rating".

Solution:

$$\begin{aligned}
E = \{ &\text{ the first customer gives a 1 star rating } \} \\
&\qquad \cup \{ \text{ the second customer gives a 1 star rating } \}.
\end{aligned}$$

(b) Suppose that both customers randomly assign their ratings, giving equal probabilities to all possible star ratings. In this case, all 25 sample points have equal probability. Compute the probability of the event in part (a).

Solution: Using the additive rule,

$$\begin{aligned}
\mathrm{P}(E) &= \mathrm{P}(1 \text{ from first customer}) + \mathrm{P}(1 \text{ from second customer}) \\
&\qquad - \mathrm{P}(1 \text{ from the first customer } \textit{and } 1 \text{ from second customer}) \\
&= \tfrac{1}{5} + \tfrac{1}{5} - \tfrac{1}{25} \\
&= \tfrac{9}{25} \\
&= 36\%.
\end{aligned}$$

11. Suppose that two customers give ratings to the same restaurant on Yelp.

(a) Describe the event "the average of their ratings is 3.5 or 4".
Hint: this is the same event as "the sum of their ratings is 7 or 8."

> **Solution:** Define two events:
>
> $$\begin{aligned} S_7 &= \{ \text{ the sum of their ratings is 7 } \} \\ &= \{(2,5),(3,4),(4,3),(5,2)\}, \\ S_8 &= \{ \text{ the sum of their ratings is 8 } \} \\ &= \{(3,5),(4,4),(5,3)\}. \end{aligned}$$
>
> Then, the event we care about is $E = S_7 \cup S_8$.

(b) As in problem 10(b), suppose that both customers randomly assign their ratings with equal probability for all possible star ratings, so that all 25 sample points have equal probability. Compute the probability of the event in part (a).

> **Solution:** Using the additive rule,
>
> $$\begin{aligned} \mathrm{P}(E) &= \mathrm{P}(S_7 \cup S_8) \\ &= \mathrm{P}(S_7) + \mathrm{P}(S_8) - \mathrm{P}(S_7 \cap S_8). \end{aligned}$$
>
> We note that the sum can't be 7 and 8 simultaneously, so S_7 and S_8 are mutually exclusive events, i.e. $S_7 \cap S_8 = \emptyset$. Thus,
>
> $$\begin{aligned} \mathrm{P}(E) &= \mathrm{P}(S_7) + \mathrm{P}(S_8) \\ &= \tfrac{4}{25} + \tfrac{3}{25} \\ &= \tfrac{7}{25} \\ &= 28\%. \end{aligned}$$